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# Quivers from Matrix Factorizations

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## Abstract

We discuss how matrix factorizations offer a practical method of computing the quiver and associated superpotential for a hypersurface singularity. This method also yields explicit geometrical interpretations of D-branes (i.e., quiver representations) on a resolution given in terms of Grassmannians. As an example we analyze some non-toric singularities which are resolved by a single  $\mathbb{P}^1$  but have “length” greater than one. These examples have a much richer structure than conifolds. A picture is proposed that relates matrix factorizations in Landau–Ginzburg theories to the way that matrix factorizations are used in this paper to perform noncommutative resolutions.

# 1 Introduction

To a Gorenstein singularity in three complex dimensions can be associated a quiver together with a superpotential in two equivalent ways. To simplify discussion let us assume this singularity is isolated. First, using string theory language one may consider a stack of D3-branes at the singular point. Then, as discussed in [1, 2], one may associate a 4-dimensional field theory to the near-horizon geometry using the AdS-CFT correspondence. This four dimensional field theory is an  $N = 1$  gauge theory. The gauge group and matter content is described by a quiver and the field theory comes with a superpotential.

Alternatively, using more purely mathematical ideas, suppose the singularity admits a crepant resolution  $X$ . Then the following picture appears to be generally true. The bounded derived category  $\mathbf{D}(X)$  of  $X$  is equivalent to the bounded derived category of representations of some quiver  $Q$  with relations. The relations are given by the derivatives of a single polynomial — the superpotential.

The equivalence of these two pictures arises from the derived category description of B-type D-branes as proposed in [3] (for a review, see [4]). The superpotential arises from the  $A_\infty$  structure of the derived category as discussed, for example, in [5–7].

It should be noted that the quiver and superpotential are not unique for a given singularity. It was noted in [8] that this ambiguity is a form of Seiberg duality [9]. It arises from the fact that two quivers (and sets of relations) can yield equivalent derived categories.

The question of interest in this paper is how does one compute a quiver together with its superpotential from a given singularity. There has been considerable work devoted to this question in the past several years. A small sample of such work is given in [10–15].

A purpose here is to give a direct, mathematically rigorous method that should work for *any* hypersurface singularity of dimension 3. It should also be straight-forward to extend this method to complete intersections. The restriction of the dimension to 3 arises because it is only in this dimension that a superpotential can be associated to the quiver relations.

The general idea comes from Van den Bergh’s notion of a noncommutative crepant resolution [16, 17]. This, in turn, arises from notions in D-brane physics as observed by Berenstein and Leigh [18].

Noncommutative resolutions are related to maximal Cohen–Macaulay (MCM) modules, which, in turn, are related to matrix factorizations via Eisenbud’s work [19]. Associating matrix factorizations to hypersurface singularities is not new — it was described in [20, 21] for example. The purpose of this paper is to see if it can be practically applied to yield a quiver and relations for interesting examples.

The easiest example of a crepant resolution has a single  $\mathbb{P}^1$  as the exceptional set as analyzed in [22]. While the humble conifold is an example of such a singularity, there are many more richer examples. The idea of “length” [23] can be used as a partial classification as we will explain in section 3.1. Our examples will concern cases of length 2.

The normal bundle of a  $\mathbb{P}^1$  as the exceptional set is of the form  $\mathcal{O}(-1, -1)$ ,  $\mathcal{O}(-2, 0)$  or  $\mathcal{O}(-3, 1)$ . The latter is by far the richest and a classification of singularity types in the class is, as yet, unknown although some progress has been made in this direction [24]. We will consider two sets of such singularities and compute the quivers together with the

superpotentials.

It is also of interest to understand the precise geometry of certain basic D-branes. Of particular interest are the “fractional branes” into which a 0-brane may decay when it coincides with the singularity. In quiver language these fractional branes are the one-dimensional quiver representations. Specializing to one example we will show how to explicitly find the objects in the derived category associated to these D-branes. We will see how one of the fractional branes corresponds to a sheaf that cannot be associated to a vector bundle.

Matrix factorizations are known to describe D-branes in the context of Landau–Ginzburg theories [25]. It is very interesting to compare our use of matrix factorizations with those associated to Landau–Ginzburg theories and we will discuss a possible relation later in this paper.

## 2 Noncommutative Resolutions

### 2.1 The resolution

Let  $\mathbb{C}^4$  have coordinates  $(w, x, y, z)$  and let  $S = \mathbb{C}[w, x, y, z]$ . Let  $f \in S$  be a polynomial such that  $f = 0$  has an isolated singularity at the origin. Put

$$R = \frac{S}{(f)}, \quad (1)$$

and  $Y = \text{Spec } R$  is a singular affine variety.

Let  $M$  be a finitely generated  $R$ -module and let  $\dim(M)$  be the length of the shortest projective resolution of  $M$ . The global dimension,  $\text{gldim } R$ , is defined as the largest value of  $\dim(M)$  for all such modules.

Because  $Y$  is not smooth,  $\text{gldim } R$  is not finite. That is, there are finitely-generated  $R$ -modules  $M$  which have no finite projective resolution. According to [19], there is a minimal free resolution of such an  $M$  which takes the form

$$\cdots \longrightarrow R^{\oplus d} \xrightarrow{\Psi} R^{\oplus d} \xrightarrow{\Phi} R^{\oplus d} \xrightarrow{\Psi} R^{\oplus d} \xrightarrow{\Phi} \cdots \longrightarrow R^{\oplus n_1} \longrightarrow M. \quad (2)$$

That is, the resolution becomes asymptotically periodic with period 2. Lifting the maps  $\Psi$  and  $\Phi$  to  $S$  we have matrices with polynomial entries. These matrices yield matrix factorizations:

$$\Phi\Psi = \Psi\Phi = f \cdot \mathbf{1}. \quad (3)$$

Conversely, given a matrix factorization of the form (3), we may write  $M = \text{coker } \Psi$  to find a module which has an infinite free resolution.

Any module of the form  $M = \text{coker } \Psi$  coming from a matrix factorization is a “Maximal Cohen–Macaulay” (MCM) module. An MCM module is defined in terms of module depths but the only salient fact for our purposes is that an MCM module always arises from a matrix factorization, where we include the trivial case  $\Phi = 1$ ,  $\Psi = f$ , yielding the possibility  $M \cong R$ .

Let  $M$  be an MCM-module (corresponding to a non-trivial matrix factorization) and define

$$A_1 = \text{End}_R(R \oplus M). \quad (4)$$

$A_1$  is a noncommutative  $\mathbb{C}$ -algebra and should be considered as a non-commutative “enhancement” of the original coordinate ring

$$R = A_0 = \text{End}_R(R), \quad (5)$$

by the MCM module  $M$ .  $A_1$  may or may not have finite global dimension. If it does not, one looks for further MCM modules to produce

$$A = A_d = \text{End}_R(R \oplus M_1 \oplus M_2 \oplus \dots \oplus M_d). \quad (6)$$

Ultimately one hopes that  $A$  has finite global dimension for large enough  $d$ . When this happens one defines [17] the noncommutative resolution of  $Y$  as  $A$ .

Let  $\pi : X \rightarrow Y$  be any crepant resolution of  $Y$ . It is conjectured that  $\mathbf{D}(X)$  is then equivalent to  $\mathbf{D}(\text{mod-}A)$ , the bounded derived category of finitely generated right  $A$ -modules. It is in this sense that  $A$  represents a resolution of  $Y$ . This conjecture has been proven in [17] for the case of one-dimensional exceptional sets in dimension 3, which is the case of interest in this paper.

## 2.2 Path algebras and superpotentials

Consider the  $\mathbb{C}$ -algebra

$$A = \text{End}_R(M_0 \oplus M_1 \oplus M_2 \oplus \dots \oplus M_m), \quad (7)$$

where  $M_0 = R$ . We compose homomorphisms right to left as usual. This algebra may be viewed as the path algebra of a quiver  $Q$  with  $m$  vertices and some relations. Let  $c_j \in A$  be the idempotent element corresponding to the identity in  $\text{Hom}_R(M_j, M_j)$ . This is a path of length zero at node  $j$ . Define

$$P_j = c_j A. \quad (8)$$

This may be viewed as the space generated by all paths *ending* at node  $j$  and is a projective right  $A$ -module. If  $\gamma$  is a path from node  $i$  to node  $j$  then  $\gamma P_i \subset P_j$  and this path gives a homomorphism from node  $i$  to node  $j$ . Thus the  $M_j$ 's in the tilting module  $M_0 \oplus M_1 \oplus M_2 \oplus \dots \oplus M_m$  correspond to the  $P_j$ 's in the category of right  $A$ -modules.<sup>1</sup>

We are interested in relations of a specific type, namely those coming from a *superpotential*. Let  $A_{\text{cyc}}$  denote the quotient of a subalgebra of  $A$  generated by cycles in the quiver, where we identify cyclic permutations of arrows. The superpotential is an element

$$\mathcal{W} \in A_{\text{cyc}}, \quad (9)$$

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<sup>1</sup>There are two conventions in the literature. One is to use right  $A$ -modules. If quiver representations are viewed as left  $A$ -modules then the direction of the arrows in  $Q$  must be reversed. By using right  $A$ -modules we keep the matrix multiplication for the rest of the paper in the conventional direction!

such that the relations of the quiver are generated by the cyclic derivatives of  $\mathcal{W}$ . That is, if arrows in the quiver are denoted  $a_i$ , one defines

$$\frac{\partial}{\partial a_j} a_{i_1} a_{i_2} \dots a_{i_n} = \sum_{s, i_s=j} a_{i_{s+1}} a_{i_{s+2}} \dots a_{i_n} a_{i_1} \dots a_{i_{s-1}}, \quad (10)$$

(modifying in the obvious way if  $j = i_1$  or  $i_n$ ) and extends this by linearity to  $\mathcal{W}$ .

Let us emphasize that we are defining the superpotential in terms of the relations. There is another definition in terms of the  $A_\infty$ -algebra of the derived category  $\mathbf{D}(X)$  which is more directly associated to the physics definition. The equivalence of these definitions has been proved in some cases [5, 6]. Having determined the superpotential for a given example in terms of relations one can then verify that it agrees with the  $A_\infty$ -algebra as we discuss in section 4.3.

$A$  is an  $A$ - $A$  bimodule. A projective  $A$ - $A$  bimodule is a summand of  $A \otimes_{\mathbb{C}} A$  and so  $A$  itself is not, in general, a projective  $A$ - $A$  bimodule. Suppose we have a projective resolution

$$\dots \longrightarrow B_2 \longrightarrow B_1 \longrightarrow B_0 \longrightarrow A \longrightarrow 0. \quad (11)$$

Let  $M$  be a right  $A$ -module. It was shown in [26] that

$$\dots \longrightarrow M \otimes_A B_2 \longrightarrow M \otimes_A B_1 \longrightarrow M \otimes_A B_0 \longrightarrow M \longrightarrow 0, \quad (12)$$

is a resolution by projective right  $A$ -modules of  $M$ . Thus, if we can find a resolution of length  $d$  for (11) then the global dimension of  $A$  can be no greater than  $d$ .

In fact, [26] gives a prescription for constructing the resolution (11) up to  $B_2$ . In the special case that the relations come from a superpotential one can write down a specific candidate for a projective resolution [27]. Introduce the projective bimodules

$$P_{ij} = A c_i \otimes c_j A. \quad (13)$$

Now we have a resolution

$$\bigoplus_{v \in \Gamma_0} P_{vv} \xrightarrow{\phi_3} \bigoplus_{a \in \Gamma_1} P_{o(a), t(a)} \xrightarrow{\phi_2} \bigoplus_{a \in \Gamma_1} P_{t(a), o(a)} \xrightarrow{\phi_1} \bigoplus_{v \in \Gamma_0} P_{vv} \xrightarrow{m} A. \quad (14)$$

The maps are defined as follows.  $m$  is simply multiplication  $m : A \otimes A \rightarrow A$ . We also have

$$\phi_1(c_{t(a)} \otimes c_{o(a)}) = c_{t(a)} \otimes a - a \otimes c_{o(a)}. \quad (15)$$

The relations are in one-to-one correspondence with the arrows since a relation is obtained from  $\partial \mathcal{W} / \partial a$ . A relation  $r$  is a linear combination of paths starting at  $t(a)$  and ending at  $o(a)$ . If one such path is given by  $a_m a_{m-1} \dots a_1$  then

$$\begin{aligned} \phi_2(a_m a_{m-1} \dots a_1) &= c_{t(a_m)} \otimes a_{m-1} \dots a_1 + \\ &\sum_{i=2}^{m-1} \left( a_m a_{m-1} \dots a_{i+1} \otimes a_{i-1} \dots a_1 \right) + a_m a_{m-1} \dots a_2 \otimes c_{o(a_1)}. \end{aligned} \quad (16)$$

$\phi_2$  is then defined on relations by extending by linearity. Finally

$$\phi_3(c_v \otimes c_v) = \sum_{t(a)=v} a \otimes c_{t(a)} - \sum_{o(a)=v} c_{o(a)} \otimes a, \quad (17)$$

where the image of the summands on the right end up in the relation associated to the arrow  $a$ .

Applying the functor  $c_i \otimes_A -$  to (14) we obtain

$$P_i \xrightarrow{\psi_3} \bigoplus_{o(a)=i} P_{t(a)} \xrightarrow{\psi_2} \bigoplus_{t(a)=i} P_{o(a)} \xrightarrow{\psi_1} P_i \twoheadrightarrow L_i, \quad (18)$$

where  $L_i = c_i$  is the “vertex simple” one-dimensional representation of  $A$  associated to the vertex  $i$ . (18) is a projective resolution of  $L_i$  if (14) is exact. The maps  $\psi_1$  and  $\psi_3$  consist of appending the corresponding arrows in the sum. The map  $\psi_2$  is a matrix whose  $(i, j)$ th entry is the part of the relation  $\partial\mathcal{W}/\partial a_i$  ending with  $a_j$  with  $a_j$  removed.

Whether (14) really is an exact sequence depends on the superpotential  $\mathcal{W}$ . It was shown in [7] that this is equivalent to  $A$  being a “Calabi–Yau algebra” of dimension 3. That is,  $A$  admits something similar to a Serre functor modified for the fact that  $X$  is not compact. We refer to [7] for the details of the definition. The Serre functor corresponds to the existence of spectral flow in the associated conformal field theory and so corresponds to the fact we have a supersymmetric solution.

An analysis of the exactness of (14) may be performed by using noncommutative Gröbner basis methods as described in [28], for example. Let  $\mathbf{P}$  be the path algebra before the relations are imposed and let  $I$  be the double-sided ideal of relations. One needs to find a Gröbner basis,  $\mathcal{G}$ , for  $I$ . This can be attempted using methods described in [28] but there is no guarantee that the resulting basis will have a finite number of elements. Indeed, for the examples in this paper  $\mathcal{G}$  is not finite. However, given a systematic presentation of  $\mathcal{G}$  one can proceed to analyze (14).

The first step in analyzing a Gröbner basis is a choice of monomial ordering on  $\mathbf{P}$ . We use the length-right-lexicographic ordering. First a choice of ordering on the vertices and on the arrows of the quiver is made with all vertices less than all arrows. Paths are then ordered by length. Then, for paths of equal length we use a lexicographic ordering reading right-to-left using the chosen order on the arrows (and vertices for zero-length paths).

This monomial ordering on the algebra  $\mathbf{P}$  can be extended to an ordering on the  $\mathbf{P}$ -modules in the resolution (14) following [26]. Consider the  $\bigoplus_{a \in \Gamma_1} P_{t(a), o(a)}$  term first. Suppose

$$\begin{aligned} q \otimes p &\in P_{t(a_i), o(a_i)} \\ s \otimes r &\in P_{t(a_j), o(a_j)}. \end{aligned} \quad (19)$$

We then impose an ordering by the following sequence of rules

1.  $l(q) < l(s)$
2.  $l(p) > l(r)$

3.  $q < s$
4.  $p > r$
5.  $a_i > a_j$ .

In each, the subsequent rule is applied if the preceding inequality test results in equality. The same ordering is applied for  $\bigoplus_{a \in \Gamma_0} P_{vv}$  except the last test is irrelevant. The key property required is that

$$\text{LT}(x) > \text{LT}(y) \Rightarrow \text{LT}(\phi_n(x)) > \text{LT}(\phi_n(y)), \quad \forall n. \quad (20)$$

This is immediately true for  $\phi_3$  and  $\phi_1$ . By choosing an appropriate modification of rule 5 for  $\bigoplus_{a \in \Gamma_1} P_{o(a),t(a)}$  depending on the details of the relations, it can be made true for  $\phi_2$ .

**Theorem 1** *The complex (14) is exact if and only if the complexes associated to the vertex simples (18) are exact for all nodes in the quiver.*

This is proved as follows. We already know that (18) is exact if (14) is exact from the exactness of  $M \otimes_A -$ . To prove that (14) is exact we need to show that  $\ker(\phi_2) \subset \text{im}(\phi_3)$  and  $\ker(\phi_1) \subset \text{im}(\phi_2)$ . Suppose  $x \in \ker(\phi_2)$ . From (20) we know that  $\text{LT}(\phi_2(x))$  cannot cancel with any other term in the image of  $x$ . If  $\text{LT}(x) = q \otimes p$  then we know from the ordering that  $\text{LT}(\phi_2)$  is of the form  $q \otimes zp$  for some path  $z$  arising from the relations. But this is exactly the part of the map  $\phi_2$  which is seen by the resolutions of the vertex simples (18). That is, if all the resolutions (18) are exact it must be that  $zp \in I$  implies that  $p$  may be reduced (with respect to the ordering) modulo terms in the image of  $\psi_3$ . Thus  $x$  may be reduced modulo the image of  $\phi_3$ . Thus, by induction,  $x$  is in the image of  $\phi_3$ . Thus  $\ker(\phi_2) \subset \text{im}(\phi_3)$ . The same argument shows  $\ker(\phi_1) \subset \text{im}(\phi_2)$ . ■

This yields an algorithm for computing the noncommutative resolution of a hypersurface singularity:

1. Begin with  $T = R$ .
2. Compute the quiver and relations for the path algebra  $A = \text{End}_R(T)$ .
3. If the relations which arise are compatible with a superpotential, check if the complexes (18) are all exact. If they are, we stop.
4. The global dimension of  $A$  is presumably infinite so we can find an irreducible module  $M$  with no finite projective resolution.
5. Replace  $T$  by  $T \oplus M$  and repeat from step 2.

Unfortunately, we don't have a proof that this algorithm terminates in general.

## 2.3 Maps between cokernels

One can be quite systematic about computing the quiver and looking for relations. Suppose we have two  $R$ -modules of the form  $M = \text{coker } \Psi$  and  $M' = \text{coker } \Psi'$  where  $\Psi$  is an  $n \times n$  matrix and  $\Psi'$  is an  $n' \times n'$  matrix. Any homomorphism from  $M$  to  $M'$  lifts to a homomorphism  $\alpha : R^n \rightarrow R^{n'}$ . The converse is trickier, however: a given homomorphism  $\alpha : R^n \rightarrow R^{n'}$  descends to a homomorphism  $M \rightarrow M'$  if and only if there exists  $\alpha' : R^n \rightarrow R^{n'}$  such that

$$\alpha\Psi = \Psi'\alpha'.$$

(This guarantees that the image of  $\Psi$  is mapped to the image of  $\Psi'$  by  $\alpha$ .) Note that  $\alpha'$  won't in general be unique.

Our main concern in this paper will be with a *single* matrix factorization of the form (3). We build a natural quiver with two nodes from this matrix factorization, with one node representing  $M = \text{coker } \Psi$  and the other representing the rank 1  $R$ -module  $R$  (which can be written as  $\text{coker } (0)$ ).

Homomorphisms from  $R \rightarrow M$  are given by arbitrary  $n \times 1$  matrices, and we can use as a generating set for these the standard basis (column) vectors for  $R^n$ : namely,  $e_1, e_2, \dots, e_n$ . However, if a homomorphism's image is contained in the image of  $\Psi$  (i.e., in the column space of the matrix), then the homomorphism is trivial

Homomorphisms from  $M \rightarrow R$  are given by  $1 \times n$  matrices  $\alpha$  such that  $\alpha\Psi = 0$ ; a basis for those is given by the rows of  $\Phi$ , which we can represent in the form  $\alpha_i = e_i^T \Phi$ .

In most cases, there will be additional endomorphisms of  $M$  or of  $R$  which need to be added to this quiver (and which may allow us to eliminate some of the  $\alpha_i$  and  $e_j$ ), but for the moment, we consider the algebra  $\mathcal{B}$  generated by the maps  $\alpha_i$  and  $e_j$ . The task is to find the abstract relations which these maps satisfy, and to determine whether the resulting algebra is the full endomorphism algebra of  $R \oplus M$  or not.

There is one type of relation in the algebra  $\mathcal{B}$  which is always present: these are relations derived from the fact that endomorphisms of  $R$  are given by  $1 \times 1$  matrices, so they must commute. That is,

$$\alpha_i e_j \alpha_k e_\ell = \alpha_k e_\ell \alpha_i e_j \tag{21}$$

for all  $i, j, k, \ell$ . Note that  $\alpha_i e_j$  is represented by the  $1 \times 1$  matrix  $[\Phi_{ij}]$ .

The next three types of relations are derived from two specific properties which the given matrix factorization may have. First, if there are linear relations among the matrix entries  $\Phi_{ij}$ , then the corresponding relations among the  $\alpha_i e_j$ 's must hold in the algebra  $\mathcal{B}$ . Second, if every entry in  $\Psi$  is a linear combination of entries of  $\Phi$ , then we get a relation from each row of  $\Psi$  by the following construction. Since  $e_i^T \Psi = \sum_j \Psi_{ij} e_j^T$ , we see

$$0 = e_i^T \Psi \Phi = \sum_j \Psi_{ij} e_j^T \Phi = \sum_j \Psi_{ij} \alpha_j.$$

Writing

$$\Psi_{ij} = \sum_{k,\ell} c_{ij}^{k\ell} \Phi_{k\ell} = \sum_{k,\ell} c_{ij}^{k\ell} \alpha_k e_\ell,$$



we find a relation

$$0 = \sum_{j,k,\ell} c_{ij}^{k\ell} \alpha_k e_\ell \alpha_j$$

for each  $i$ . And third, again when every entry in  $\Psi$  is a linear combination of entries of  $\Phi$ , we get a relation from each column of  $\Psi$  as follows. We have  $\Psi e_j = \sum_i e_i \Psi_{ij}$  and this combination is the zero homomorphism  $R \rightarrow M$  since its image lies in the column space of  $\Psi$ . Thus,

$$0 = \sum_i e_i \Psi_{ij} = \sum_{i,k,\ell} e_i (c_{ij}^{k\ell} \Phi_{k\ell}) = \sum_{i,k,\ell} c_{ij}^{k\ell} e_i \alpha_k e_\ell.$$

## 3 Easy Examples

### 3.1 Length

Let  $M$  be an  $R$ -module. Let us assume both  $R$  and  $M$  are Noetherian. A *composition series* of length  $n$  for  $M$  is a strictly decreasing chain of submodules

$$M = M_0 \supset M_1 \supset \dots \supset M_n = 0, \quad (22)$$

such that  $M_j/M_{j+1}$  is nonzero and has no nonzero proper submodules for all  $j$ .

It can be shown that all compositions series for a fixed  $M$  have the same length (see, for example, [29]) and this defines the *length* of  $M$ . Similarly one can define the length of coherent sheaves.

Now consider a resolution of singularities

$$\pi : X \rightarrow Y, \quad (23)$$

where  $Y = \text{Spec } R$  has an isolated singularity at  $p$ . Let  $\mathfrak{m}$  denote the maximal ideal in  $R$  associated to the point  $p$ . The inclusion map  $\mathfrak{m} \rightarrow R$  pulls back via  $\pi^*$  to a map of sheaves<sup>2</sup>

$$f : \pi^* \mathfrak{m} \rightarrow \mathcal{O}_X. \quad (24)$$

Let  $\mathcal{E}$  denote the sheaf on  $X$  associated to  $\text{coker } f$ .  $E$ , the support of  $\mathcal{E}$ , is then the exceptional set of the resolution.  $E$  decomposes into a union of irreducible components:

$$E = \bigcup_i E_i. \quad (25)$$

Correspondingly we have a primary decomposition of the module associated to  $\mathcal{E}$ :

$$\mathcal{E} = \bigcap_k M_k. \quad (26)$$

Let  $M_i$  be an element of the above primary decomposition corresponding to a minimal associated prime of  $\mathcal{E}$ . Then  $M_i$  is associated to one of the components  $E_i \subset E$ . We then have

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<sup>2</sup>If  $M$  is an  $R$ -module we will also use  $M$  to denote the associated coherent sheaf on  $\text{Spec } R$ .

**Definition 1** *The length of an irreducible component  $E_i \subset E$  of the exceptional set is defined as the length of the module  $M_i$ .*

One may regard the length as the multiplicity of a component of the exceptional set for a resolution.

In this paper our interest concerns resolutions with a single  $\mathbb{P}^1$  as the exceptional set. Such resolutions are therefore labeled by a single length.

Cases where lengths of greater than one are very common but generally harder to analyze. For example, suppose that the resolution  $\pi : X \rightarrow Y$  is described torically. That is, it is obtained by some subdivision of a fan  $\Sigma$ . The explicit form of such a resolution in terms of coordinate patches (see, for example, [30]) immediately implies that only components of length one can ever appear in the exceptional set. Similarly any generic deformation of a toric case will still only exhibit length one. Thus, the various toric methods that can apply for computing quivers, etc., cannot be used in any direct way to study lengths greater than one.

### 3.2 Du Val singularities

A useful and straight-forward example of higher lengths concerns the well-known ADE resolutions of  $\mathbb{C}^2/G$ , for  $G \subset \mathrm{SL}(2, \mathbb{Z})$  as studied in the McKay correspondence in [31–33]. In particular, for the current context, see [34, 35]. Here, the lengths are the Kac labels on the Dynkin diagram. E.g., the resolution associated to  $E_6$  has lengths

$$\begin{array}{c} \textcircled{2} \\ | \\ \textcircled{1} - \textcircled{2} - \textcircled{3} - \textcircled{2} - \textcircled{1} \end{array} \quad (27)$$

The description of these Du Val singularities in terms of noncommutative resolutions and matrix factorizations is implicit in [32] and described in [36].

Let  $G$  act on  $\mathbb{C}^2 = \mathrm{Spec}(\mathbb{C}[s, t])$  and thus  $\mathbb{C}[s, t]$ .  $Y = \mathbb{C}^2/G$  then has coordinate ring  $R = \mathbb{C}[s, t]^G$ . Denote by  $\rho_i$  the nontrivial irreducible representations of  $G$ . There is then a decomposition

$$\mathbb{C}[x, y] = R \oplus \left( \bigoplus_i M_i \otimes \rho_i \right), \quad (28)$$

where  $M_i$  are  $R$ -modules. All such  $M_i$  can be associated to matrix factorizations as listed in [32, 33],

$$M_i = \mathrm{coker} \Psi_i, \quad (29)$$

where  $\Psi_i$  is a  $2l_i \times 2l_i$  matrix with entries in  $R$ . It turns out, from the classification in [32], that each  $M_i$  is associated to an irreducible component of the exceptional divisor with length  $l_i$ .

A noncommutative resolution of  $\mathbb{C}^2/G$  is then described by the path algebra of the McKay quiver which is also given by

$$A = \text{End}_R \left( R \oplus \left( \bigoplus_i M_i \right) \right). \quad (30)$$

### 3.3 A Flop

Consider the hypersurface singularity

$$x^2 - y^{2k} + wz = 0, \quad (31)$$

where  $k$  is a positive integer. This admits a  $2 \times 2$  matrix factorization

$$\Psi = \begin{pmatrix} w & -x - y^k \\ x - y^k & z \end{pmatrix}, \quad \Phi = \begin{pmatrix} z & x + y^k \\ -x + y^k & w \end{pmatrix}. \quad (32)$$

Set  $M = \text{coker } \Psi$ . The quiver for  $A = \text{End}_R(R \oplus M)$  is constructed as follows.

First note that a given morphism  $f \in \text{Hom}_R(M_1, M_2)$  can be multiplied by any  $r \in R$  giving  $\text{Hom}_R(M_1, M_2)$  the structure of an  $R$ -module. For the cases considered this module is finitely-generated and can be computed straight-forwardly using packages such as Macaulay 2. One obtains

$$\text{Hom}_R(R, R) \cong \text{Hom}_R(M, M) \cong R, \quad (33)$$

and both  $\text{Hom}(R, M)$  and  $\text{Hom}(M, R)$  are  $R$ -modules with two generators. These generators are precisely the  $e_i$ 's and  $\alpha_j$ 's of section 2.3. Thus we account for all paths if we include the  $e_i$ 's,  $\alpha_j$ 's and all maps corresponding to multiplication by  $x, y, z$  and  $w$ .

It is easy to see that

$$\begin{aligned} \alpha_1 e_1 &= z \\ \alpha_1 e_2 &= x + y^k \\ \alpha_2 e_1 &= -x + y^k \\ \alpha_2 e_2 &= w, \end{aligned} \quad (34)$$

and so all maps  $R \rightarrow R$  are generated except for multiplication by  $y$  in the case  $k > 1$ . We also have

$$e_1 \alpha_1 = \begin{pmatrix} z & x + y^k \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} \pmod{\text{Im } \Psi} \quad (35)$$

and similarly maps from  $M \rightarrow M$  corresponding to multiplication by  $w, x + y^k$  and  $-x + y^k$  can be constructed. Thus, if  $k > 1$  we need to add two arrows to the quiver to account for multiplication by  $y$ . The resulting quiver is

$$\quad (36)$$

If  $k = 1$  these two extra loops are not added to the quiver. There are now some obvious relations. Assuming  $k > 1$  we have

$$\begin{aligned}
\alpha_1 e_2 + \alpha_2 e_1 &= 2y_1^k \\
e_2 \alpha_1 + e_1 \alpha_2 &= 2y_2^k \\
e_i y_1 &= y_2 e_i \\
y_1 \alpha_i &= \alpha_i y_2.
\end{aligned} \tag{37}$$

One can check these relations imply all the relations discussed in section 2.3. Furthermore, it is clear these relations can be integrated up to form a superpotential

$$\mathcal{W} = y_1 \alpha_1 e_2 + y_1 \alpha_2 e_1 - \alpha_1 y_2 e_2 - \alpha_2 y_2 e_1 - \frac{2}{k+1} y_1^{k+1} + \frac{2}{k+1} y_2^{k+1}. \tag{38}$$

This recovers the known result of [11, 37]. Similarly the case  $k = 1$  recovers the simple conifold quiver and relations of [1, 2].

### 3.4 A generalized conifold

Consider the hypersurface singularity

$$f = x^3 - xy^2 - wz = 0. \tag{39}$$

Let

$$M_1 = \text{coker} \begin{pmatrix} x & -z \\ -w & x^2 - y^2 \end{pmatrix}, \tag{40}$$

corresponding to a matrix factorization of  $f$ . Now try the path algebra  $A = \text{End}(R \oplus M_1)$ . This yields a quiver

$$\begin{array}{ccc}
& \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \\
& \downarrow & \\
& \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \\
& \downarrow & \\
& \begin{pmatrix} w & x \end{pmatrix} & \\
& \downarrow & \\
& \begin{pmatrix} x^2 - y^2 & z \end{pmatrix} & \\
\end{array}
\tag{41}$$

Some relations correspond to the fact that  $y$  commutes with paths of length 2 going from  $R$  to  $M_1$  and back again. It is easy to see that it is impossible for these relations to come from the derivatives of any superpotential. Correspondingly, the global dimension of  $A$  is infinite.

The quiver (41) is therefore a *partial* noncommutative resolution of the singularity (39). There is another matrix factorization yielding

$$M_2 = \text{coker} \begin{pmatrix} x^2 + xy & -z \\ -w & x - y \end{pmatrix}. \tag{42}$$

Now set  $A = \text{End}(R \oplus M_1 \oplus M_2)$ . Then we obtain a quiver

$$\begin{array}{c}
 & R & \\
 \nearrow \gamma_1 & & \nwarrow \alpha_1 \\
 M_2 & & M_1 \\
 \nwarrow \gamma_2 & & \nearrow \alpha_2 \\
 & & \\
 \xleftarrow{\beta_2} & & \xrightarrow{\beta_1}
 \end{array}
 \tag{43}$$

where

$$\begin{aligned}
 \alpha_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \alpha_2 &= \begin{pmatrix} w & x \end{pmatrix} \\
 \beta_1 &= \begin{pmatrix} x+y & 0 \\ 0 & 1 \end{pmatrix} & \beta_2 &= \begin{pmatrix} 1 & 0 \\ 0 & x+y \end{pmatrix} \\
 \gamma_1 &= \begin{pmatrix} x-y & z \end{pmatrix} & \gamma_2 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}
 \end{aligned}
 \tag{44}$$

There are obvious relations such as  $\gamma_1\gamma_2\alpha_1\alpha_2 = \alpha_1\alpha_2\gamma_1\gamma_2$  but these cannot arise directly from a superpotential since there is no loop at  $R$ . So we should look for relations between paths of length 3. Writing down all such paths gives relations arising from the superpotential

$$\mathcal{W} = \alpha_1\alpha_2\alpha_1\alpha_2 + \beta_1\beta_2\beta_1\beta_2 + \gamma_1\gamma_2\gamma_1\gamma_2 + \alpha_2\alpha_1\gamma_1\gamma_2 + \beta_2\beta_1\alpha_1\alpha_2 + \gamma_2\gamma_1\beta_1\beta_2, \tag{45}$$

in agreement with [38]. This is the complete resolution.

## 4 Flops of Length Two

### 4.1 The universal flop of length two

A threefold singularity  $x \in X$  which has a crepant resolution  $\pi : Y \rightarrow X$  such that  $\pi^{-1}(x)$  is a rational curve of length two is always part of a *flop* [23]: there is a second crepant resolution  $\pi : Y^+ \rightarrow X$  of the same singularity.

A “universal flop of length two”  $\mathcal{X}$  was constructed in [36]. This is a hypersurface in  $\mathbb{C}^7$  which has two small resolutions  $\mathcal{Y} \rightarrow \mathcal{X}$  and  $\mathcal{Y}^+ \rightarrow \mathcal{X}$  determined from a matrix factorization by a “Grassmann blowup.” This hypersurface is universal in the sense that for every threefold singularity  $X$  which has a crepant resolution  $Y \rightarrow X$  with exceptional set a rational curve of length two, there is a map from  $X$  to  $\mathcal{X}$  such that  $Y \rightarrow X$  is the pullback of  $\mathcal{Y} \rightarrow \mathcal{X}$ . In particular, all of the algebraic properties of the hypersurface in  $\mathbb{C}^7$  will be inherited by any such threefold via the map  $X \rightarrow \mathcal{X}$ .

We now give a partial analysis of  $\text{End}(R \oplus M)$  for the module  $M$  obtained from the matrix factorization associated to the universal flop of length two. The generators and relations we find will be a common feature of *any* flop of length two. In later sections we will specialize to specific instances of the universal flop, which produce specific flops, and we will refine the generators and relations of the algebra in those cases.

The hypersurface equation in  $\mathbb{C}^7$  which describes the universal flop of length two is

$$f = x^2 + uy^2 + 2vyz + wz^2 + (uw - v^2)t^2. \quad (46)$$

Put  $\Psi = xI + \Xi$  and  $\Phi = xI - \Xi$  where

$$\Xi = \begin{bmatrix} -vt & y & z & t \\ -uy - 2vz & vt & -ut & z \\ -wz & wt & -vt & -y \\ -uwt & -wz & uy + 2vz & vt \end{bmatrix}.$$

The entries in  $\Phi$  all occur as the entries in  $1 \times 1$  matrices in our algebra, since  $\alpha_i e_j = e_i^T \Phi e_j = [\Phi_{ij}]$  (with  $i$  and  $j$  fixed). There are linear relations among these: essentially, some duplicates among the entries, up to sign. When duplicate entries occur, we get a relation among our operators  $\alpha_i$  and  $e_j$ . This gives the following relations:

$$\begin{aligned} \alpha_1 e_1 &= \alpha_3 e_3 & \alpha_1 e_2 &= -\alpha_3 e_4 & \alpha_1 e_3 &= \alpha_2 e_4 \\ \alpha_2 e_1 &= -\alpha_4 e_3 & \alpha_2 e_2 &= \alpha_4 e_4 & \alpha_3 e_1 &= \alpha_4 e_2 \end{aligned} \quad (47)$$

There are four relations which we get from the columns of  $\Psi$ :  $\sum_j \Psi_{ij} \alpha_j = 0$ . Using the fact that each  $\Psi_{ij}$  coincides with some  $\Phi_{k\ell} = \alpha_k e_\ell$ , we can write these relations as

$$\begin{aligned} 0 &= \alpha_4 e_4 \alpha_1 + \alpha_3 e_4 \alpha_2 - \alpha_2 e_4 \alpha_3 - \alpha_1 e_4 \alpha_4 \\ 0 &= \alpha_4 e_3 \alpha_1 + \alpha_3 e_3 \alpha_2 - \alpha_2 e_3 \alpha_3 - \alpha_2 e_4 \alpha_4 \\ 0 &= -\alpha_4 e_2 \alpha_1 - \alpha_3 e_2 \alpha_2 + \alpha_4 e_4 \alpha_3 - \alpha_3 e_4 \alpha_4 \\ 0 &= -\alpha_4 e_1 \alpha_1 - \alpha_4 e_2 \alpha_2 - \alpha_4 e_3 \alpha_3 + \alpha_3 e_3 \alpha_4 \end{aligned} \quad (48)$$

where we used the relations in eq. (47) to simplify these slightly.

There are also relations from the columns of  $\Psi$ , taking the form  $\sum_i e_i \Psi_{ij} = 0$ . Again using the fact that each  $\Psi_{ij}$  coincides with some  $\Phi_{k\ell} = \alpha_k e_\ell$ , we can write these relations as

$$\begin{aligned} 0 &= e_1 \alpha_4 e_4 + e_2 \alpha_4 e_3 - e_3 \alpha_4 e_2 - e_4 \alpha_4 e_1 \\ 0 &= e_1 \alpha_3 e_4 + e_2 \alpha_3 e_3 - e_3 \alpha_3 e_2 - e_4 \alpha_4 e_2 \\ 0 &= -e_1 \alpha_2 e_4 - e_2 \alpha_2 e_3 + e_3 \alpha_4 e_4 - e_4 \alpha_4 e_3 \\ 0 &= -e_1 \alpha_1 e_4 - e_2 \alpha_2 e_4 - e_3 \alpha_3 e_4 + e_4 \alpha_3 e_3 \end{aligned} \quad (49)$$

where again we used the relations in eq. (47) to simplify.

There are at least two more endomorphisms of  $M$  which so far have not been accounted

for.<sup>3</sup> We denote these

$$a = a' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -u & 0 & 0 & 0 \\ -2v & 0 & 0 & 1 \\ 0 & 2v & -u & 0 \end{bmatrix}$$

$$b = b' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \end{bmatrix}$$

(Recall that for an endomorphism of  $M$ , lifted to a map  $E : R^4 \rightarrow R^4$ , we needed a second map  $E'$  such that  $E\Psi = \Psi E'$ .) Note that for both of the above endomorphisms, since  $\Phi = 2xI - \Psi$ , we also have  $\Phi E = E\Phi$ .

These endomorphisms have two important properties (which easily follow from  $\Phi E = E\Phi$ ). First, letting

$$c = [x + vt \quad -y \quad -z \quad -t]$$

be the first row of  $\Phi$ , the rows of  $\Phi$  are  $\alpha_1 = c$ ,  $\alpha_2 = \alpha a$ ,  $\alpha_3 = \alpha b$ , and  $\alpha_4 = -\alpha ab$ . Second, letting

$$d = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

be the fourth standard basis (column) vector of  $R^4$ , the standard basis vectors are  $e_1 = bac$ ,  $e_2 = -bc$ ,  $e_3 = ac$ , and  $e_4 = c$ .

Thus, with these endomorphisms, our quiver simplifies tremendously: we only need one arrow in each direction ( $c$  and  $d$ , respectively) together with two loops  $a$  and  $b$  at the node corresponding to  $M$ . There is also a possibility that more loops are required at the  $R$  node as we see in section 4.4.

Let us see how the relations that we already found translate to this language. The relations in eq. (47) are all tautologies, as is easily verified. The number of operators  $\alpha_i e_j$  which need to be considered for eq. (21) is smaller, and consists of the ten operators

$$\begin{aligned} cd &= \alpha_1 e_4, & cad &= \alpha_2 e_4, & cbd &= \alpha_3 e_4, & ca^2 d &= \alpha_2 e_3, \\ cabd &= -\alpha_4 e_4, & cbad &= \alpha_3 e_3, & cb^2 d &= -\alpha_3 e_2, \\ cabad &= -\alpha_4 e_3, & cab^2 d &= \alpha_4 e_2, & cab^2 ad &= -\alpha_4 e_1. \end{aligned}$$

There are 45 relations which assert that these all commute (for example:  $cdcad = cadcd$ ).

---

<sup>3</sup>We found these by making detailed calculations in the full deformation of the D4 quiver which had been determined in [36].

Finally, the relations in eq. (48) can be written as:

$$\begin{aligned}
0 &= c(-abdc + bdca - adcb + dcab) \\
0 &= c(-abadc + badca - aadcb + adcab) \\
0 &= c(-bbadc + bbdca - abdcb + bdcab) \\
0 &= c(abbadc - abbdca + abadcb - badcab)
\end{aligned}$$

while those in eq. (49) can be written as

$$\begin{aligned}
0 &= (-bbdcab + bdcaba - adcabb + dcabba) d \\
0 &= (bbdcb - bdcba + adcb - dcabb) d \\
0 &= (-badca + bdcaa - adcab + dcaba) d \\
0 &= (-badc + bdca - adcb + dcba) d
\end{aligned}$$

There is no reason to suppose that the universal flop can be described in terms of superpotential since it is of dimension  $> 3$ . In order to proceed further it is easier to specialize to particular cases.

## 4.2 The Morrison–Pinkham example

The first examples of flops of length 2 were given by Laufer [22, Example 2.3]; a deformation of Laufer’s first example was subsequently found by Morrison and Pinkham [39, Example 10]. Later, Reid [40, Example (5.15)] pointed out that these examples could be put in a standard form, and it was shown in [36] that there is a corresponding matrix factorization. We give the matrix factorization for the Morrison–Pinkham example explicitly.

The Morrison–Pinkham example has equation

$$x^2 + y^3 + wz^2 + w^3y - \lambda wy^2 - \lambda w^4 = 0, \quad (50)$$

where  $\lambda \in \mathbb{C}$  is a parameter. Although this might appear, at first sight, to be a continuous family of singularities as one varies  $\lambda$ , there are actually only two isomorphism classes of singularities in the sense of differential equivalence [41]. These cases are  $\lambda = 0$  and  $\lambda \neq 0$ .

This maps to the universal flop of length two given in (46) via

$$\begin{aligned}
t &= -w \\
u &= y - \lambda w \\
v &= 0
\end{aligned} \quad (51)$$

giving the matrix factorization

$$\Psi := \begin{pmatrix} x & y & z & -w \\ -(y - \lambda w)y & x & (y - \lambda w)w & z \\ -wz & -w^2 & x & -y \\ (y - \lambda w)w^2 & -wz & (y - \lambda w)y & x \end{pmatrix} \quad (52)$$



$$\Phi := \begin{pmatrix} x & -y & -z & w \\ (y - \lambda w)y & x & -(y - \lambda w)w & -z \\ wz & w^2 & x & y \\ -(y - \lambda w)w^2 & wz & -(y - \lambda w)y & x \end{pmatrix}. \quad (53)$$

In this case, the matrix entries of  $\Phi$  generate the ring

$$R = \mathbb{C}[w, x, y, z]/(x^2 + y^2 + wz^2 + w^3y - \lambda wy^2 - \lambda w^4),$$

so we don't need any additional loops at the node  $R$ .

All of the relations from the previous section hold. In addition, we get some new ones, derived from our new relations (51). But perhaps it is easiest just to directly write down the relations among loops at  $R$  in this case. The former ten operators are now reduced to four:

$$cd = \alpha_1 e_4, \quad cad = \alpha_2 e_4, \quad cbd = \alpha_3 e_4, \quad cbad = \alpha_3 e_3$$

and these must pairwise commute:

$$\begin{aligned} cdcad &= cadcd \\ cdcbd &= cbdcd \\ cdcbad &= cbadcd \\ cadcbd &= cbdcad \\ cadcbad &= cbadcad \\ cbdcbad &= cbadcbd \end{aligned}$$

The remaining six operators now give relations (from  $vt$ ,  $uy + 2vz$ ,  $ut$ ,  $wz$ ,  $wt$ ,  $uwt$ , respectively):

$$\begin{aligned} 0 &= c(ab + ba)d \\ cabad &= c(bdcb - \lambda dcb)d \\ ca^2d &= c(-bdc + \lambda dc)d \\ cab^2d &= c(-adc)d \\ cb^2d &= c(-dc)d \\ -cab^2ad &= c(-bdcdc + \lambda dcdc)d \end{aligned} \quad (54)$$

Explicit computation using the matrices directly shows that *all* these relations reduce to

$$\begin{aligned} (b^2 + dc)d &= 0 \\ c(b^2 + dc) &= 0 \\ ab + ba &= 0 \\ a^2 + bdc + dcb + \lambda b^2 + b^3 &= 0. \end{aligned} \quad (55)$$

It is now a simple observation that our master relations (55) are obtained as derivatives of the superpotential<sup>4</sup>

$$\mathcal{W} = b^2dc + \frac{1}{2}dcdc + a^2b + \frac{1}{3}\lambda b^3 + \frac{1}{4}b^4. \quad (56)$$

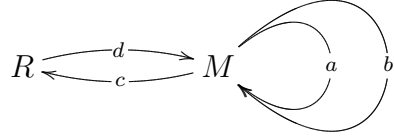
Note that the endomorphisms at  $M$  comprising elements of  $R$  are accounted for by

$$\begin{aligned} w &= -b^2 \\ x &= dcba + badc + b^3a \\ y &= -\lambda b^2 - a^2 \\ z &= -dca - adc - ab^2 \end{aligned} \quad (57)$$

It is then a simple matter to use Macaulay 2 to check that we have accounted for all possible endomorphisms of  $R \oplus M$  by explicitly computing  $\text{End}(R \oplus M)$  as an  $R$ -module.

We now have the following:

**Theorem 2** *For the relations given by (55), the quiver associated to the noncommutative resolution of (50) is given by*



$$\quad (58)$$

and we have an associated superpotential (56).

Now that we have a candidate quiver and superpotential we need to check we have completed the noncommutative resolution, that is, the global dimension is finite.

Recall that  $\mathbf{P}$  is the path algebra of this quiver without relations imposed and  $I$  is the ideal of relations. We need to impose an admissible order on the monomials of  $\mathbf{P}$  and then find a Gröbner basis for  $I$ . Picking an order at random tends to produce an infinite Gröbner basis.

Let  $\mathbb{C}(\lambda)$  be the field of rational functions in  $\lambda$ . We may view the path algebra as an algebra over this field. The ideal of relations is then quasi-homogeneous if we assign weight 2 to  $b, c, d$ , and  $\lambda$ ; and weight 3 to  $a$ . This allows us to form a weight-lexicographic order on  $\mathbf{P}$  by using weight and then arrow order  $a > b > c > d$ . The result is that we have a finite

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<sup>4</sup>Some aspects of this superpotential were also derived in [42] using the methods of [11].

Gröbner basis for the relations of the form:<sup>5</sup>

$$\begin{aligned}
& ab + ba \\
& cb^2 + cdc \\
& b^2d + dcd \\
& a^2 + b^3 + \lambda b^2 + bdc + dcb \\
& adcd + b^2ad \\
& adcb - badc + \lambda b^2a - 2b^3a - bdca - dcba \\
& cbdc - cdcdb.
\end{aligned}$$

Let  $L_R$  denote the one-dimensional representation of  $A$ . Using (18) we have a candidate resolution:

$$R \xrightarrow{d} M \xrightarrow{b^2+dc} M \xrightarrow{c} R \longrightarrow L_R \quad (59)$$

**Theorem 3** *The complex (59) is exact.*

We already know this complex is exact except possibly at the first two terms from section 2.2. To check exactness at the first term we need to show that left-multiplication by  $d$  is injective. Given our choice of term ordering, with  $d$  the smallest arrow, any nontrivial element of the ideal of relations of the form  $dx$  would be detected by a Gröbner basis element left-divisible by  $d$ . There is no such element and so the complex is exact at the first term.

Exactness at the second term can be argued as follows. Suppose, in  $\mathbf{P}$ ,  $z = (b^2 + dc)x$  is in the ideal  $I$ . Thus, the leading term  $b^2x$  must be divisible by a leading term of  $\mathcal{G}$ . Without loss of generality we may assume  $x$  has already been reduced with respect to  $\mathcal{G}$ . Inspection of  $\mathcal{G}$  shows that the only possibilities are  $\text{LT}(x) = bdy$  or  $\text{LT}(x) = dy$ .

Suppose  $\text{LT}(x) = bdy$ . Then  $z$  may be reduced by subtracting  $b(b^2d + dcd)y$ . The problem now is that  $bdc dy$  cannot be canceled by anything remaining in  $(b^2 + dc)x$  and it cannot be reduced as a leading term. So we cannot reduce  $z$  to lie in  $I$ .

So it must be that  $\text{LT}(x) = dy$ . Then  $z$  is reduced by subtracting  $(b^2 + dc)dy$  to leave  $z'$ . Now repeat this process until  $z$  is reduced to 0. Since the leading term of  $x$  at each stage was of the form  $dy$  we require  $x$  to be in the image of left-multiplication by  $d$ . Thus (59) is exact.

Alternatively one may prove exactness at the second term as follows. Use the Gröbner basis above to compute the Hilbert functions for  $R$  and  $M$  by enumerating all possible paths. One can show that

$$\begin{aligned}
H_R(q) &= \frac{1 + q^5}{(1 - q^2)(1 - q^4)(1 - q^7)} \\
H_M(q) &= \frac{1 + q^3}{(1 - q^2)^2(1 - q^7)}.
\end{aligned} \quad (60)$$

---

<sup>5</sup>We thank Ed Green for a useful correspondence on this. We also used the computer package “bergman”, developed by Jörgen Backelin, for computations for the Gröbner basis.

We do not include the details here as they are lengthy. One can then use this to show that the complex (59) is exact given that it is exact at all terms except one. ■

If  $L_M$  is the one-dimensional representation associated to the  $M$  node we have a candidate complex quasi-isomorphic to  $L_M$ :

$$M \xrightarrow[f_2]{\begin{pmatrix} c \\ a \\ b \end{pmatrix}} R \oplus M^{\oplus 2} \xrightarrow[f_1]{\begin{pmatrix} cd & 0 & cb \\ 0 & b & a \\ bd & a & dc + \lambda b + b^2 \end{pmatrix}} R \oplus M^{\oplus 2} \xrightarrow[f_0]{(d \ a \ b)} M \quad (61)$$

**Theorem 4** *The complex (61) is exact.*

We need to prove  $\ker(f_1) \subset \text{im}(f_2)$ . The fact  $\ker(f_0) \subset \text{im}(f_1)$  proves that if

$$ax + by = 0, \quad (62)$$

then  $x = bz$  and  $y = az$  for some path  $z$ . Suppose  $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$  is in  $\ker(f_1)$ . It follows that  $b = az$  and  $c = bz$  for some path  $z$ . This, in turn implies

$$bd(a - cz) = 0. \quad (63)$$

We already know that left-multiplication by  $d$  is injective. To prove that left-multiplication by  $b$  is injective we need a new ordering where the arrow  $b$  is less than all other arrows. All possibilities of the remaining ordering leaves an infinite Gröbner basis. For example if we order  $b < d < c < a$  then all elements of  $\mathcal{G}$  of weight  $\geq 10$  can be shown to be of the form

$$cb^{2n}dc + cb^{2n+2}. \quad (64)$$

While this cannot be directly exhibited by computer, it is easy enough to apply the algorithm in [28] by hand to verify this. Again, no element of  $\mathcal{G}$  is left-divisible by  $b$ . It follows that  $a = cz$  and we prove exactness of (61) at the second term.

Since we have shown above that left-multiplication by  $b$  is injective we have that (61) is exact at the first term. ■

Finally let us recall that we only expect there to be two inequivalent possibilities; namely  $\lambda = 0$  or  $\lambda \neq 0$ . Also recall that the superpotential is only defined up to  $A_\infty$ -isomorphisms. These  $A_\infty$ -isomorphisms correspond to nonlinear reparametrizations of  $a, b, c, d$  as discussed in [11]. Suppose we have a conformal field theory associated with the AdS/CFT correspondence for this singularity. As such, the superpotential should be quasi-homogeneous. All this implies that if  $\lambda \neq 0$ , the term  $\frac{1}{4}b^4$  in the superpotential is *irrelevant* and may be removed by an  $A_\infty$ -isomorphism. Thus we expect only two inequivalent superpotentials:

$$\mathcal{W} = \begin{cases} b^2dc + \frac{1}{2}dcdc + a^2b + \frac{1}{4}b^4 & \text{for } \lambda = 0 \\ b^2dc + \frac{1}{2}dcdc + a^2b + \frac{1}{3}b^3 & \text{for } \lambda \neq 0. \end{cases} \quad (65)$$

### 4.3 $A_\infty$ Structure

Let  $\mathbf{D}(\text{fdmod-}A)$  be the full subcategory of  $\mathbf{D}(\text{mod-}A)$  given by finite-dimensional quiver representations.  $\mathbf{D}(\text{fdmod-}A)$  has an  $A_\infty$ -structure coming from projective resolutions of vertex simple objects  $L_i$  as was discussed in [5, 6]. (We refer to [7, 43] for an more in-depth discussion of  $A_\infty$ -algebras and quivers.) The  $A_\infty$ -structure encodes the superpotential. There has been some progress towards equating the superpotential arising from the quiver relations and the superpotential arising from this  $A_\infty$  algebra [44], but to date a complete proof seems elusive.

It is a simple matter to verify that the latter superpotential coincides with that calculated above. Let us put  $\lambda = 0$  to simplify the discussion.

Each arrow in the quiver from node  $i$  to node  $j$  is associated to a basis element of  $\text{Ext}^1(L_i, L_j)$ . We represent these arrows as maps between the projective resolutions. For example,  $a$  is given by

$$\begin{array}{ccccccc}
 & & M & \xrightarrow{\begin{pmatrix} c \\ a \\ b \end{pmatrix}} & R \oplus M^{\oplus 2} & \xrightarrow{f_1} & R \oplus M^{\oplus 2} \xrightarrow{\begin{pmatrix} d & a & b \end{pmatrix}} M \\
 & & \downarrow \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 1 & 0 \end{pmatrix} \\
 M & \xrightarrow{\begin{pmatrix} c \\ a \\ b \end{pmatrix}} & R \oplus M^{\oplus 2} & \xrightarrow{f_1} & R \oplus M^{\oplus 2} & \xrightarrow{\begin{pmatrix} d & a & b \end{pmatrix}} & M
 \end{array} \quad (66)$$

while  $b$  is given by

$$\begin{array}{ccccccc}
 & & M & \xrightarrow{\begin{pmatrix} c \\ a \\ b \end{pmatrix}} & R \oplus M^{\oplus 2} & \xrightarrow{f_1} & R \oplus M^{\oplus 2} \xrightarrow{\begin{pmatrix} d & a & b \end{pmatrix}} M \\
 & & \downarrow \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 0 & c \\ 0 & 1 & 0 \\ d & 0 & b \end{pmatrix} & & \downarrow \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\
 M & \xrightarrow{\begin{pmatrix} c \\ a \\ b \end{pmatrix}} & R \oplus M^{\oplus 2} & \xrightarrow{f_1} & R \oplus M^{\oplus 2} & \xrightarrow{\begin{pmatrix} d & a & b \end{pmatrix}} & M
 \end{array} \quad (67)$$

We now form a differential graded algebra with basis given by the generators of  $\text{Ext}^1(L_i, L_j)$  as above. The product structure is given by composing maps while one may define a differential  $d$  in the usual way on chain maps. For example, the composition  $b \cdot b$  is a chain map yielding an element of  $\text{Ext}^2(L_M, L_M)$ . This element is zero since it is associated to chain

homotopy given by a map we call  $\gamma$ :

$$\begin{array}{ccccccc}
M & \xrightarrow[\begin{smallmatrix} f_2 \\ \begin{pmatrix} c \\ a \\ b \end{pmatrix} \end{smallmatrix}]{} & R \oplus M^{\oplus 2} & \xrightarrow[\begin{smallmatrix} f_1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{smallmatrix}]{} & R \oplus M^{\oplus 2} & \xrightarrow[\begin{smallmatrix} f_0 \\ (d \ a \ b) \end{smallmatrix}]{} & M \\
\downarrow \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} & & \downarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} & & \downarrow (0 \ 0 \ 0) & & \\
M & \xrightarrow[\begin{smallmatrix} f_2 \\ \begin{pmatrix} c \\ a \\ b \end{pmatrix} \end{smallmatrix}]{} & R \oplus M^{\oplus 2} & \xrightarrow[\begin{smallmatrix} f_1 \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{smallmatrix}]{} & R \oplus M^{\oplus 2} & \xrightarrow[\begin{smallmatrix} f_0 \\ (d \ a \ b) \end{smallmatrix}]{} & M
\end{array} \tag{68}$$

Thus we say that  $b \cdot b = \mathbf{d}\gamma$ .

One can now follow exactly the procedure in [11] to extract an  $A_\infty$ -algebra from this dga and thus find a superpotential. The result is that one recovers (56) and so the two definitions of superpotential agree.

#### 4.4 Laufer's examples

Now consider the examples described in [22]. Start with an integer  $n > 0$ , and the equation<sup>6</sup>

$$x^2 + y^3 + wz^2 + w^{2n+1}y = 0. \tag{69}$$

We can map this to the universal flop of length two via:

$$\begin{aligned}
t &= w^n \\
u &= y \\
v &= 0
\end{aligned}$$

Then the matrix factorization is

$$\Psi := \begin{pmatrix} x & y & z & w^n \\ -y^2 & x & -yw^n & z \\ -wz & w^{n+1} & x & -y \\ -yw^{n+1} & -wz & y^2 & x \end{pmatrix} \tag{70}$$

$$\Phi := \begin{pmatrix} x & -y & -z & -w^n \\ y^2 & x & yw^n & -z \\ wz & -w^{n+1} & x & y \\ yw^{n+1} & wz & -y^2 & x \end{pmatrix} \tag{71}$$

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<sup>6</sup>Laufer presents this with different variables. In these variables, the equation reads:  $v_4^2 + v_2^3 - v_1v_3^2 - v_1^{2n+1}v_2 = 0$  (correcting a typo from [22]).

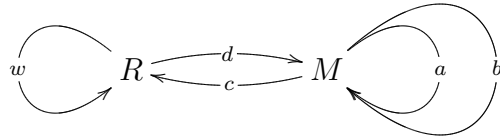
The maps in the quiver are:

$$\begin{aligned}
a &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -y & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -y & 0 \end{pmatrix} \\
b &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -w & 0 & 0 & 0 \\ 0 & w & 0 & 0 \end{pmatrix} \\
c &= (x \quad -y \quad -z \quad -w^n) \\
d &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
\end{aligned} \tag{72}$$

The endomorphisms in  $R$  at  $M$  correspond to

$$\begin{aligned}
w &= -b^2 \\
x &= dcba + badc + (-1)^m b^{2n+1} a \\
y &= -a^2 \\
z &= -dca - adc - (-1)^m ab^{2n}
\end{aligned} \tag{73}$$

An extra loop is required at  $R$  if  $n > 1$ . Let's call it  $w$  as it amounts to multiplication by  $w$ . This gives a quiver:



$$\tag{74}$$

The superpotential is then

$$\mathcal{W} = dwc + dc b^2 + a^2 b + \frac{w^{n+1}}{n+1} + \frac{(-1)^n}{2n+2} b^{2n+2}. \tag{75}$$

Confirmation that this noncommutative resolution is complete is similar to section 4.2. Note that the superpotential is quasi-homogeneous which facilitates computations of Gröbner bases, etc.

In the case that  $n = 1$  we do not require the loop associated to  $w$  and the quiver becomes that of (58). The superpotential becomes

$$\mathcal{W} = dcb^2 - \frac{1}{2}cdcd + a^2b - \frac{1}{4}b^4, \quad (76)$$

which is equivalent to the Morrison–Pinkham example with  $\lambda = 0$  as expected.

## 4.5 A Resolution

So far we have made no reference to any resolution of the singularities in question. In each case there are two inequivalent resolutions related by  $x \mapsto -x$ . These two resolutions lead to a *flop* between the two possibilities.

A resolution can be described purely in terms of a matrix factorization itself as described in [36]. In general, suppose we have a hypersurface  $X \subset \mathbb{C}^N$  described by  $f = 0$  with isolated singularity at the origin. Suppose further we have an  $n \times n$  matrix factorization  $\Psi\Phi = f$ . Away from the origin, on  $X_{\text{smooth}}$ , the rank of  $\Phi$  is  $r$ . In all the cases studied in this paper  $n$  is even and  $r = n/2$ . The kernel of  $\Phi$  defines a point in the Grassmannian  $\text{Gr}(r, n)$ . The Grassmannian resolution,  $\tilde{X}$ , of  $f = 0$  is then defined as the *closure* of the point-set

$$\{(x, v) | x \in X_{\text{smooth}}, v \in \ker \Phi_x\} \subset \mathbb{C}^N \times \text{Gr}(r, n). \quad (77)$$

The space  $\text{Gr}(r, n)$  (assuming  $n = 2r$ ) is of dimension  $r^2$  and is covered by  $\binom{n}{r}$  affine charts using the Plücker coordinates. For example, suppose  $n = 4$ . In chart  $U_0$  we could consider matrices

$$J_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \quad (78)$$

where  $\Phi J_0 = 0$  and in chart  $U_1$  we could consider matrices

$$J_1 = \begin{pmatrix} 1 & 0 \\ \beta_{11} & \beta_{12} \\ 0 & 1 \\ \beta_{21} & \beta_{22} \end{pmatrix} \quad (79)$$

where  $\Phi J_1 = 0$ . The transition functions between these charts are given by

$$J_0 \begin{pmatrix} 1 & 0 \\ \beta_{11} & \beta_{12} \end{pmatrix} = J_1. \quad (80)$$

Some lengthy algebra, described in [36]<sup>7</sup> can be used to describe the coordinate charts on the blowup.

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<sup>7</sup>Note that [36] describes the blowup in terms of  $\ker \Psi$  rather than  $\ker \Phi$ . This effectively just changes the sign of  $x$ .



Suppose we fix on the Laufer examples from section 4.4. We will recover the exact description of the charts used by Laufer. To facilitate this comparison we will need to switch coordinates to a new set we denote by hats. In chart  $U_1$  one gets the equations (from equations labeled by  $\mu_3$ ,  $\mu_{1,2}$ , and  $\mu_{3,2}$  respectively in [36]):

$$\begin{aligned}\beta_{2,2}^2 + y + \beta_{1,2}^2 w &= 0 \\ \beta_{1,2} y + z + \beta_{2,2} w^n &= 0 \\ \beta_{1,2} w^{n+1} - x - \beta_{2,2} y &= 0\end{aligned}\tag{81}$$

These can be solved:

$$\begin{aligned}y &= -\beta_{2,2}^2 - \beta_{1,2}^2 w \\ z &= \beta_{1,2} \beta_{2,2}^2 + \beta_{1,2}^3 w - \beta_{2,2} w^n \\ x &= \beta_{1,2} w^{n+1} + \beta_{2,2}^3 + \beta_{1,2}^2 \beta_{2,2} w\end{aligned}\tag{82}$$

Now relabel the coordinates of  $\mathbb{C}^4$  by  $w = -v_1$ ,  $x = (-1)^n v_4$ ,  $y = v_2$  and  $z = -v_3$ . If we assign  $v_1 = \hat{z}_1$ ,  $\beta_{2,2} = (-1)^{n+1} \hat{z}_2$  and  $\beta_{1,2} = \hat{w}$ , then these solutions become

$$\begin{aligned}v_1 &= \hat{z}_1 \\ v_2 &= -\hat{z}_2^2 + \hat{w}^2 \hat{z}_1 \\ v_3 &= -\hat{w} \hat{z}_2^2 + \hat{w}^3 \hat{z}_1 - \hat{z}_1^n \hat{z}_2 \\ v_4 &= -\hat{w} \hat{z}_1^{n+1} - \hat{z}_2^3 + \hat{w}^2 \hat{z}_1 \hat{z}_2,\end{aligned}\tag{83}$$

where  $(\hat{w}, \hat{z}_1, \hat{z}_2)$  are good affine coordinates for the resolution of  $X$  in the patch  $U_1$ , and  $(v_1, v_2, v_3, v_4)$  are coordinates of the embedding  $X \subset \mathbb{C}^4$ . Note that  $v_4^2 + v_2^3 - v_1 v_3^2 - v_1^{2n+1} v_2 = 0$  which is the equation for  $X$ .

To go to patch  $U_0$  we define new coordinates  $\hat{x} = \alpha_{12}$  and  $\hat{y}_2 = (-1)^{n+1} \alpha_{22}$ . We need a third good coordinate  $\hat{y}_1$  which is a little awkward to spot. The trick is to note that we need to correctly parametrize the branch of  $X$  where  $v_1 = v_2 = v_4 = 0$ . To this end put  $v_3 = \hat{y}_1 + \xi$ , where  $\xi$  is a function of  $(\hat{x}, \hat{y}_1, \hat{y}_2)$  which vanishes when  $\hat{x} = \hat{y}_2 = 0$ . A little algebra yields a solution

$$\begin{aligned}\hat{z}_1 &= \hat{x}^3 \hat{y}_1 + \hat{y}_2^2 + \hat{x}^2 \hat{y}_2^{2n+1} \\ \hat{z}_2 &= \hat{x}^{-1} \hat{y}_2 \\ \hat{w} &= \hat{x}^{-1}\end{aligned}\tag{84}$$

which is exactly Laufer's set of transition functions. Note that we manifestly see that the normal bundle of the exceptional  $\mathbb{P}^1$  is of type  $(-3, 1)$ . Note also that  $\tilde{X}$  is *not* the total space of this bundle.

## 4.6 Geometrical Interpretation of Branes

Now that we have the explicit from of the resolution, it is interesting to see the geometric interpretation of some basic objects in the D-brane category.

Our tilting objects are  $R$  and  $M$ . Once we have an interpretation of these in terms of the derived category of coherent sheaves then we can build any other object we desire.

$R$  is obvious enough — it gives the structure sheaf  $\mathcal{O}_{\tilde{X}}$  on the resolution.  $M$  also has a clear interpretation given the explicit form of the Grassmannian blow-up as we now explain.

For definiteness let us focus on the Laufer example where  $n = 1$  and look at the affine coordinate patch  $U_1$  with coordinates  $(\hat{w}, \hat{z}_1, \hat{z}_2)$ . The matrix  $J_1$  given in (79) is

$$\begin{pmatrix} 1 & 0 \\ -\hat{z}_2 & \hat{w} \\ 0 & 1 \\ -\hat{w}\hat{z}_1 & \hat{z}_2 \end{pmatrix} \quad (85)$$

and has image equal to  $\ker(\Phi) = \text{Im}(\Psi)$ . Thus,  $M$  which is defined as the cokernel of  $\Psi$  can therefore be viewed in the patch  $U_1$  as the cokernel of the matrix  $J_1$ . Viewed thus as an  $R$ -module we may find the coherent sheaf  $\mathcal{M}$  associated to  $M$ . The matrix (85) has constant rank 2 and so  $\mathcal{M}$  is a locally free sheaf of rank 2 in  $U_1$ . Similarly  $M$  has constant rank in patch  $U_0$  and so  $\mathcal{M}$  is locally free of rank 2 over the entire resolution.

So, given any object in the derived category of quiver representations, we may find the corresponding geometric object in the derived category of coherent sheaves on the resolution by first writing a free resolution in terms of the projective objects  $R$  and  $M$  and then replacing these objects by  $\mathcal{O}_{\tilde{X}}$  and  $\mathcal{M}$  respectively.

Of particular interest are the two simplest one-dimensional quiver representations  $L_R$  and  $L_M$ . In physics terminology these are often called the “fractional branes”. We have the projective resolutions of these objects in (59) and (61).<sup>8</sup>

Thus, for example, the object  $L_M$  can be represented as the complex

$$M \xrightarrow[f_2]{\begin{pmatrix} c \\ a \\ b \end{pmatrix}} R \oplus M^{\oplus 2} \xrightarrow[f_1]{\begin{pmatrix} -cd & 0 & cb \\ 0 & b & a \\ bd & a & dc-b^2 \end{pmatrix}} R \oplus M^{\oplus 2} \xrightarrow[f_0]{\begin{pmatrix} d & a & b \end{pmatrix}} M. \quad (86)$$

But now this is a complex of modules over a *commutative* algebra  $R = \mathbb{C}[\hat{w}, \hat{z}_1, \hat{z}_2]$  and we may use computer algebra packages such as Macaulay 2 to determine the cohomology.

Let us number positions in the complex so that the final  $M$  is in position 0. One can compute that the cohomology groups  $H^i$  vanish in (86) for  $i = -3, -2, 0$  and

$$H^{-1} = R/(\hat{z}_1, \hat{z}_2). \quad (87)$$

It is worth emphasizing the distinction here. As a complex of modules over the noncommutative path algebra  $A$ , the complex (86) is exact except in position 0 and there the cohomology is  $L_M$ . However, as a complex of modules over  $\mathbb{C}[\hat{w}, \hat{z}_1, \hat{z}_2]$  it is exact everywhere except at position  $-1$ .

One may perform the same computation in the other coordinate patch  $U_0$  and the result is

$$H^{-1} = R/(\hat{y}_1, \hat{y}_2). \quad (88)$$

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<sup>8</sup>A few sign changes are required to convert between the Morrison–Pinkham form and the Laufer form.

The fact that the cohomology is concentrated in a single position in the complex implies that this is equivalent, in the derived category, to a complex with a single nonzero module in one position. It follows that the geometrical interpretation of  $L_M$  is  $\mathcal{F}[1]$ , where  $\mathcal{F}$  is a sheaf supported on the exceptional curve  $C$  (where  $\hat{z}_1 = \hat{z}_2 = 0$ ) where it is locally free and rank one. The “[1]” denotes a shift left in the derived category. Since a locally free sheaf of rank one on  $C$  must be of the form  $\mathcal{O}_C(a)$  for some  $a \in \mathbb{Z}$ , we have that  $L_M$  corresponds to  $\mathcal{O}_C(a)[1]$ . We determine  $a$  shortly.

A similar computation for  $L_R$  yields

$$H^0 = \begin{cases} R/(\hat{y}_1, \hat{y}_2^2) & \text{in } U_0 \\ R/(\hat{z}_1, \hat{z}_2^2) & \text{in } U_1 \end{cases} \quad (89)$$

with all other cohomologies vanishing. It follows that  $L_R$  corresponds to some sheaf  $\mathcal{L}_R$  which is an extension

$$0 \longrightarrow \mathcal{O}_C(b) \longrightarrow \mathcal{L}_R \longrightarrow \mathcal{O}_C(c) \longrightarrow 0, \quad (90)$$

for some integers  $b, c$ .

The integers  $a, b, c$  can be determined by using the equivalence between the derived category of quiver representations and the derived category of coherent sheaves. For example, for quiver representations it is easy to show that

$$\text{Ext}_A^k(R, L_M) = 0, \quad \text{for all } k. \quad (91)$$

This translates into the statement that

$$\text{Ext}_{\mathcal{O}_{\tilde{X}}}^{k+1}(\mathcal{O}_{\tilde{X}}, \mathcal{O}_C(a)) = 0, \quad \text{for all } k, \quad (92)$$

which, in turn, implies  $H^k(C, \mathcal{O}_C(a)) = 0$  for all  $k$ . This implies  $a = -1$ .

Similarly we know

$$\text{Ext}_A^0(R, L_R) = \text{Ext}_A^1(L_R, L_M) = \text{Ext}_A^1(L_M, L_R) = \mathbb{C}, \quad \text{etc.} \quad (93)$$

These conditions imply that  $b = -1$  and  $c = 0$  and that the extension (90) is nontrivial. One can compute

$$\text{Ext}_{\mathcal{O}_{\tilde{X}}}^1(\mathcal{O}_C, \mathcal{O}_C(-1)) = \mathbb{C}, \quad (94)$$

which shows that the nontrivial extension (90) is unique. This completes the proof of the following:

**Theorem 5** *If  $A$  is the path algebra of the quiver (58) subject to constraints given by the superpotential (76) then there is an equivalence of triangulated categories given by the functor*

$$\mathbf{T} : \mathbf{D}(\text{mod-}A) \rightarrow \mathbf{D}(\tilde{X}), \quad (95)$$

such that

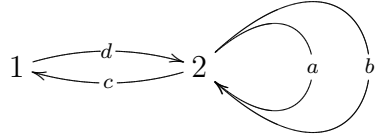
$$\begin{aligned}
\mathsf{T}(R) &= \mathcal{O}_{\tilde{X}} \\
\mathsf{T}(M) &= \mathcal{M} \\
\mathsf{T}(L_R) &= \mathcal{L} \\
\mathsf{T}(L_M) &= \mathcal{O}_C(-1)[1],
\end{aligned} \tag{96}$$

where  $\mathcal{L}$  is a sheaf corresponding to the unique nontrivial extension

$$0 \longrightarrow \mathcal{O}_C(-1) \longrightarrow \mathcal{L} \longrightarrow \mathcal{O}_C \longrightarrow 0. \tag{97}$$

It is perhaps worth emphasizing the fact that if we restrict the geometry to  $C$ , then there is no nontrivial extension of the form (97) since  $\mathrm{Ext}_{\mathcal{O}_C}^1(\mathcal{O}_C, \mathcal{O}_C(-1)) = 0$ . Thus,  $\mathcal{L}$  cannot be considered to be a locally free sheaf on  $C$  pushed forward into  $\tilde{X}$  by the inclusion map. That is, there is no description of the object  $L_R$  in terms of vector bundles on  $C$ .

Now consider the skyscraper sheaf,  $\mathcal{O}_p$  of a point  $p \in \tilde{X}$ , also known as a 0-brane. Given a tilting collection of locally sheaves of rank  $n_0, n_1, \dots$ , it was shown in [45] that  $\mathcal{O}_p$  corresponds to a quiver with dimension vector  $(n_0, n_1, \dots)$ . This means that the skyscraper sheaf on  $\tilde{X}$  is given by a quiver representation of the form



$$\tag{98}$$

where the numbers represent the dimension of the corresponding vector space. In physics language, this is a  $U(1) \times U(2)$  quiver gauge theory.

In terms of K-theory classes this implies that  $[\mathcal{O}_x] = [L_R] + 2[L_M]$ . This is entirely consistent with the identifications of the fractional branes in theorem 5. The moduli space of stable representations of the form (98) should yield the complete space  $\tilde{X}$ . We will not attempt to derive this here.

Similarly one can see that the quiver representation with dimension vector  $(1, 1)$  corresponds to  $\mathcal{O}_C$  and accordingly has no (unobstructed) deformations.

## 5 Resolutions versus Landau–Ginzburg Theories

In this paper we have utilized matrix factorizations in analyzing D-branes, i.e., the derived category of coherent sheaves, on the resolution of a hypersurface singularity. This is not, of course, the first time that matrix factorizations have appeared in the context of D-branes. It was shown in [46] that D-branes in a Landau–Ginzburg theory with 2-dimensional superpotential  $W$  are described by a category whose objects are matrix factorizations of  $W$ . In this section we give some speculative comments about the possible relation between these two appearances of matrix factorizations.

Let  $f = 0$  be a hypersurface singularity in  $\mathbb{C}^n$  and let  $R = \mathbb{C}[x_1, \dots, x_n]/(f)$  and  $X = \text{Spec } R$ . The category of D-branes in a Landau–Ginzburg theory was argued in [47] to be a quotient

$$\mathbf{D}_{\text{Sg}}(X) = \frac{\mathbf{D}(X)}{\mathfrak{P}\text{erf}(X)}, \quad (99)$$

where  $\mathfrak{P}\text{erf}(X)$  is the subcategory of  $\mathbf{D}(X)$  given by bounded complexes of locally-free sheaves of finite type. This category depends only on local information of  $X$  “inside” the singularity as explained in [47]. This is entirely consistent with the physics of a Landau–Ginzburg theory where the classical vacua are given by the critical points of  $f$  and so we expect all physics to be localized at these critical points in some sense.

The category  $\mathbf{D}_{\text{Sg}}(X)$  is triangulated but the homological grading is  $\mathbb{Z}_2$ -valued. This is to be expected since  $\mathbf{D}_{\text{Sg}}(X)$  is not a Calabi–Yau category. By taking an orbifold of a Landau–Ginzburg theory one can restore a full  $\mathbb{Z}$ -grading [48] and obtain a Calabi–Yau category but this is not desired here.

A string theory or conformal field theory associated with a  $\sigma$ -model with target space  $X$  is quite distinct from a Landau–Ginzburg theory on  $\mathbb{C}^n$  with worldsheet superpotential  $f$ . We propose one should think of the  $\sigma$ -model as seeing the space “outside” any singularity and the Landau–Ginzburg theory as seeing the space “inside”. Indeed, a singularity in  $X$  for the  $\sigma$ -model is a bad thing — the conformal field theory will be singular if we consider deforming a smooth defining equation for  $f$  to the singular case.

We propose, therefore, that the procedure of a noncommutative resolution should be viewed as moving D-branes from “inside” the singularity as seen by the Landau–Ginzburg model to the “outside” as seen by the  $\sigma$ -model. Initially, before the resolution, it is only the MCM module  $R$  itself that lives on the outside. Note that  $R$  itself, corresponding to the matrix factorization  $f = 1 \cdot f$  is a trivial D-brane for the Landau–Ginzburg theory. Now, to perform the resolution we pass the MCM modules into the tilting module. This makes the D-branes of singularity visible to the outside  $\sigma$ -model. At the same time, since the singularity is resolved, the Landau–Ginzburg theory itself becomes trivial. Thus, all that was inside the singularity has passed to the outside.<sup>9</sup>

Whether any specific physics can be attached to this description remains to be seen but it at least forms a nice picture of the relation between matrix factorizations for Landau–Ginzburg theories and matrix factorization for resolutions.

## 6 Discussion

Our analysis of these length two singularities is not complete. In particular we have not considered the exact nature of the flop. In the case of length one conifolds, one can show that the flop is associated with a symmetry of the quiver obtained by exchanged the two nodes. Since there is no such symmetry in the length two case, something must change. It would also be interesting to study the moduli space of the sky-scraper sheaf quiver (98).

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<sup>9</sup>Some similar ideas have been formulated in [49].

Another interesting question concerns massless D-branes on the singularity. On a simple conifold, either of the two vertex-simple fractional branes,  $\mathcal{O}_C$  or  $\mathcal{O}_C(-1)[1]$ , become massless (depending on the  $B$ -field) as  $C$  is blown down. What happens in the cases studied in this paper? Can the length two fractional brane  $L_R$  become massless? The usual way to analyze this would be to find some local mirror picture and find some Picard–Fuchs system, or use something like the monodromy ring of [50]. We do not know how to do this for the cases at hand.

Additionally one can consider extremal transitions. The Milnor numbers for the singularities we consider are quite large. For example, the Milnor number of the Laufer case (69) is given by  $6k + 5$ . This means the extremal transition is associated to one deformation of Kähler form (blowing up  $C$ ) and  $6k + 5$  deformations of complex structure. It would be interesting to see if such a local transition could be seen in a compact Calabi–Yau threefold.

Obviously one may also try to apply our methods to higher length cases. Whether our method provides any particular computational efficiency over other methods may be a matter of taste. However, it does provide a nice picture of how D-branes pass between Landau–Ginzburg theories and  $\sigma$ -models when one resolves the singularity.

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